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Abstract

We show that the definition of a projective coordinate frame within a Laguerre-Forsyth scheme, leads to the extension of the factorized diffeomorphism algebra. The quantum improvement of this symmetry can be performed only if these coordinates switch, at the quantum level, into a non-commutative regime.

PACS number : 11.10 - 11.25 hf

Keywords : Field Theoretical models, W-algebras, Noncommutativity.

CPT–2004/P.050

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1 Introduction

The realization of symmetries in physical models reveals sometimes unusual aspects which lead to new insights. Therefore we must go into the matter more thoroughly in order to get a clearer idea of the situation.

Usually the first step in this investigation amounts to studying the realization in the linear approximation, and in many cases this approximation fully solves the problems, with the aid of Lie group theory.

Anyhow, sometimes, there is no physical reason for the symmetry to be linear, and, on the other hand, it was realized, in the eighties, the restriction to the linear approximation was too narrow from a physical point of view.

Indeed the developments of solvable models, string theory, and the explosion of the non-linear sciences lead to a change of opinion in the researchers community.

In particular, in the investigations on the infinite-dimensional algebras, the exigency of the non linearity became quite appealing [1].

These implications were transferred into the context of string theory, and both to the theory of integrable systems and the theory of two dimensional critical phenomena [2], [3][4][5]

These new non-linear symmetries lead in particular to the so-called $\mathcal{W}$ algebras, which were interpreted as the generalizations of the well-known Virasoro algebra by allowing to introduce higher spin fields. Recall that the Virasoro algebra is the infinite dimensional Lie algebra associated with the conformal symmetries in two dimensional space-time.

One profitable approach which broadens the common knowledge of this subject is the Drinfeld-Sokolov construction [6]. It allows to derive the $\mathcal{W}$ algebras staring from affine Lie algebras.

Many other powerful approaches exist as well [7][8][9] [10][11][12][13] [14] [15]$\cdots$ In particular, our own point of view consists in an attempt in constructing $\mathcal{W}$-algebras in terms of space-time transformations [16] [16][17] [18][19]. In the present paper we suggest an improvement which, according to our opinion, could be more fruitful from the physical viewpoint, namely in order to construct some models.

For this purpose we follow the guidelines given by Forsyth [20] concerning differential equations that we shall adapt to the one dimensional complex case in order to incorporate the conformal geometry of Riemann surfaces.

The former is a textbook which, according to our opinion, is the precursor of this subject and steers our treatment.

All the mathematical details of our method will be treated elsewhere [21]: we summarize here only the main steps, since the aim of the paper is to raise some physical consequences which are involved in the present approach.

In this optic, the main result of the paper concludes that the coordinate frames defined within the Forsyth-Laguerre formalism, and the dynamics of which is given by a Lagrangian model, turn to have at the Quantum level, a non commutative regime. Of course, at the Classical one, they display the customary commutative behaviour. This conclusion derives from a counterterm required for compensating the $\mathcal{W}$-anomaly. This quantum mechanism awards the Forsyth-Laguerre frames with very unfamiliar quantum interaction properties.

In Section 2 the reader is introduced to the already well known (for the people engaged in these fields) Forsyth-Laguerre projective formalism. In particular we shall show that non-infinitesimal holomorphic changes of frames produce a special projective coordinate property (we shall call the D.O.R. mechanism), generate an algebra.

In Section 3 we discuss the B.R.S. formulation of the symmetry implemented by the previous transformations, giving rise to a finite $\mathcal{W}$ algebra.

Furthermore we shall show that the D.O.R. mechanism is naturally encapsulated in the realization of a $\mathcal{W}$-symmetry.

Anyhow, we shall see, in the formalism there is a degeneracy of a geometrical nature which does not allow to define the coordinates in a `not` unique way. This depends on the fact that the arbitrary order of derivatives of more than one coordinate shows the same coefficients expansion generating the D.O.R. mechanism.

So we are in presence of orbits of frames with identical symmetry properties. This degeneracy is well expected from the physical point of view thus showing the **full** equivalence of frames. But it might be dangerous if we want to improve the symmetry at the quantum level. In fact, from a dynamical point of view, any degeneracy could lead to some instabilities under radiative corrections.

To discuss and to solve this pathology we construct, in Section 4, a Lagrangian model for the dynamics of the Forsyth-Laguerre frames. At the quantum level anomalies occur, and consequently a symmetry breakdown.

By, using the usual methods of gauge theory, we shall show that a topological Chern-Simons counterterm allows the anomaly cancellations. The price to pay is the introduction of new mutual interactions between the coordinate fields.

In this new panorama the symmetry is restored at the quantum level, but the Laguerre-Forsyth frames must be considered as noncommutative coordinates for higher order frames.

2 The Laguerre-Forsyth frames and the D.O.R. mechanism

This section is devoted to remind the definition of the Laguerre-Forsyth projective frames and to clarify their properties.

The starting point is to consider a (two dimensional) Riemann surface with local complex coordinates $(z,\overline{z})$ and where is defined the well-known differential equation of order s [22]:

$$
\begin{aligned}
&L_s f^{(R)}(z,\overline{z}) = 0 \\
&L_s = \sum_{j=0}^{s} a^{(s)}_{(s-j)}(z,\overline{z})\partial_j, \qquad a^{(s)}_{(0)}(z,\overline{z}) = 1, \qquad a^{(s)}_{(1)}(z,\overline{z}) = 0.
\end{aligned} \tag{2.1}
$$

The previous equation admits s solutions $f^{(R)}(z,\overline{z})$ which are scalar densities under holomorphic change of charts. with conformal weight $\frac{1-s}{2}$. On the other hand, under the same change the operators $L_r(z,\overline{z})$ scale as:

$$
L_r((w,\overline{w})) = (w')^{-\frac{(1+r)}{2}} L_r(z,\overline{z})(w')^{\frac{1-r}{2}} \tag{2.2}
$$

.

If we introduce the ratios:

$$
Z^{(R)}(z,\overline{z}) = \frac{f^{(R+1)}(z,\overline{z})}{f^{(1)}(z,\overline{z})}, \qquad R = 1\cdots s-1 \tag{2.3}
$$

we construct a map from the $(z,\overline{z})$ plane to the projective $\mathbf{CP}^{(s-1)}$ space, after getting a set of $s-1$ independent scalar functions.

So we can state:

Conjecture 2.1 *In each point of the $(z,\overline{z})$ surface we can construct an $s-1$ projective dimensional frame.*

$$
\vec{Z}(z,\overline{z}) \equiv \left(Z^{(1)}, Z^{(2)}, \cdots, Z^{(s-1)}\right)(z,\overline{z}) \tag{2.4}
$$

It is trivial to remark that the $Z^{(R)}(z,\overline{z})$ "coordinates" are scalar objects with respect $(z,\overline{z})$ covariance, and the (R) index represents only a label to distinguish each "coordinates". Such frames will be called "Forsyth-Laguerre frames " in the sequel.

This gave rise to a wide class of investigations in the subject on algebraic geometry [23] [24] and higher orders uniformization of Rieman surfaces [25][26][27].

We reopen in Ref [21] this problem at a rather lower level, focusing our attention on the related properties of the Forsyth-Laguerre coordinates.

Using Eqs (2.3)(2.1) we can get, first of all, for each class of $Z^{(R)}(z,\overline{z})$, the remarkable property:

Guess 2.1

$$\begin{aligned} \partial_{(m)} Z^{(R)}(z,\overline{z}) &= \sum_{l=1}^{s-1} \mathcal{R}^{(l)}_{(m)}(z,\overline{z}) \varpi^{(R)}_{(l)}(z,\overline{z}) \\ \mathcal{R}^{(l)}_{(m)}(z,\overline{z}) &= \delta^{(l)}_{(m)} \qquad if \qquad 1 \leq m \leq s-1 \end{aligned} \tag{2.5}$$

where we have introduced the wronskian matrix:

$$\varpi(z,\overline{z}) \equiv \begin{pmatrix} \partial Z^{(1)}(z,\overline{z}) & \cdots & \partial Z^{(s-1)}(z,\overline{z}) \\ \vdots & \ddots & \vdots \\ \partial_{(s-1)} Z^{(1)}(z,\overline{z}) & \cdots & \partial_{(s-1)} Z^{(s-1)}(z,\overline{z}) \end{pmatrix} \tag{2.6}$$

This decomposition is universal for all the $Z^{(R)}(z,\overline{z})$ coordinates, so $\mathcal{R}^{(l)}_{(s+m)}(z,\overline{z})$ does not depend on the order (R) of the coordinate $Z^{(R)}(z,\overline{z})$.

Guess(2.5) states that the wronskian is a basis for its arbitrary derivatives with coefficients depending on $(z,\overline{z})$.

We shall call this mechanism as Derivative Order Reduction, and we shall denote in future as D.O.R.

The proof is trivial by direct computation, and we get:

$$\begin{aligned} \mathcal{R}^{(j)}_{(s)}(z,\overline{z}) \equiv -\frac{1}{f^{(1)}(z,\overline{z})} \Bigg[\Bigg(\sum_{l=j}^{s-1} \begin{pmatrix} l \\ j \end{pmatrix} a^{(s)}_{(s-l)}(z,\overline{z}) \partial_{l-j} f^{(1)}(z,\overline{z}) \Bigg) \\ + \begin{pmatrix} s \\ j \end{pmatrix} \partial_{s-j} f^{(1)}(z,\overline{z}) \Bigg] \end{aligned} \tag{2.7}$$

where $\mathcal{R}^{(j)}_{(s)}(z,\overline{z})$ depends on the coefficients $a^{(j)}_{(s-j)}(z,\overline{z})$ and $f^{(1)}(z,\overline{z})$(and their z

derivatives); so this decomposition does not depend on the index of the test function $f^{(P)}(z,\overline{z})$.

In particular we have from Eq (2.7)(due to convention $a^{(s)}_{(1)}(z,\overline{z})=0$):

$$\mathcal{R}^{(s-1)}_{(s)}(z,\overline{z}) = -s\partial \ln f^{(1)}(z,\overline{z}) \tag{2.8}$$

The $\mathcal{R}^{(p)}_{(m)}(z,\overline{z})$ coefficients have meaningful properties: from the very definition Eq (2.5) it follows:

$$\mathcal{R}^{(p)}_{(m+n)}(z,\overline{z}) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \sum_{l=1}^{s-1} \partial_{(j)} \mathcal{R}^{(l)}_{(n)}(z,\overline{z}) \mathcal{R}^{(p)}_{(m+(l-j))}(z,\overline{z}) \tag{2.9}$$

In particular:

$$\partial \mathcal{R}^{(l)}_{(n)}(z,\overline{z}) = \mathcal{R}^{(l)}_{(n+1)}(z,\overline{z}) - \sum_{m=1}^{s-1} \mathcal{R}^{(m)}_{(n)}(z,\overline{z}) \mathcal{R}^{(l)}_{(m+1)}(z,\overline{z}) \tag{2.10}$$

If we now perform, for such Forsyth-Laguerre frames, the finite non linear holomorphic transformations (holomorphic in the sense that $\overline{z}$ is kept fixed):

$$\begin{aligned} Z^{(R)}(z,\overline{z}) \;&\longrightarrow\; Z'^{(R)}\Big(Z^{(R)}(z,\overline{z})\Big) \\ &\equiv\; Z^{(R)}(z,\overline{z}) + \sum_{l=1}^{(\infty)} \gamma^{l}(z) \partial_l Z^{(R)}(z,\overline{z}) \end{aligned} \tag{2.11}$$

the D.O.R. mechanism, after the formal resummations of the expansion parameters, shrinks to a finite number of terms in the expansion:

$$\begin{aligned} &\sum_{l=1}^{(\infty)} \gamma^{l}(z) \partial_l Z^{(R)}(z,\overline{z}) = \sum_{l=1}^{(\infty)} \gamma^{l}(z) \sum_{m=1}^{(s-1)} \mathcal{R}^{(m)}_{(l)}(z,\overline{z}) \partial_m Z^{(R)}(z,\overline{z}) \\ &\equiv \sum_{n=1}^{(s-1)} \sigma^{(n)}(z,\overline{z}) \partial_n Z^{(R)}(z,\overline{z}) \end{aligned} \tag{2.12}$$

(with $s>1$).

The composition of two transformations as in Eqs (2.11), belongs (thanks to the D.O.R. trick) to the same space of Eq (2.12):

$$
\begin{aligned}
Z^{(R)}(z,\bar{z}) \;&\longrightarrow\; Z''^{(R)}\Big(Z'^{(R)}\Big(Z^{(R)}(z,\bar{z})\Big)\Big) \\
&= Z^{(R)}(z,\bar{z}) + \sum_{l=1}^{(\infty)} \gamma^{l}(z)\partial_{l}\Big(Z^{(R)}(z,\bar{z}) + \sum_{l'=1}^{(\infty)} \gamma^{l'}(z)\partial_{l'} Z^{(R)}(z,\bar{z})\Big) \\
&= Z^{(R)}(z,\bar{z}) + \sum_{n=1}^{(s-1)} \sigma'^{(n)}(z,\bar{z})\partial_{n} Z^{(R)}(z,\bar{z}) \qquad (2.13)
\end{aligned}
$$

This fact generates an algebra.

This shows that only in the Forsyth-Laguerre frames it is possible to extend beyond the first order derivative the infinitesimal holomorphic diffeomorphisms, retaining the structure of an algebra.

As said before the aim of this work is to study, on a physical perspective, the non linear algebra coming from the transformations Eqs (2.11) (2.12), and to show that a B.R.S. treatment of the problem allows to describe not only the symmetry, but it is also clever to include the D.O.R. mechanism, which is the cornerstone of the Forsyth-Laguerre approach.

This ends the discussion of the introductory material needed to the more physical part of the paper; the remaining part wants to show to the reader the intriguing role of the Laguerre-Forsyth frames, and their related symmetries, within a Field Theoretical model.

More specifically, while at the Classical level they optimize (in a projective approach) an ordinary coordinate system. But, when the improvement at the quantum level is performed, the symmetry conservation requirement and the anomaly absorption, assign to them a fields content, and then only within a non commutative coordinate framework the physical content can be discussed.

3 B.R.S approach

The transformations (2.11) induce [21] the B.R.S transformations :

$$
\begin{aligned}
\delta_{\mathcal{W}} Z^{(R)}(z,\bar{z}) &= \sum_{l=1}^{s-1} \mathcal{K}^{(l)}(z,\bar{z})\varpi^{(R)}_{(l)}(z,\bar{z}) \\
& 1 \le r \le s-1 \qquad (3.14)
\end{aligned}
$$

The nilpotency (and the use of Eq (2.5) is again the primary requirement to get it) gives the B.R.S. variations of the ghosts:

$$\delta_{\mathcal{W}}\mathcal{K}^{(l)}(z,\overline{z}) = \sum_{m} \mathcal{K}^{(m)}(z,\overline{z})\mathcal{B}^{(l)}_{(m)}(z,\overline{z}) \tag{3.15}$$

where $\mathcal{B}(z,\overline{z})$ is a ghost one $(s-1)\times(s-1)$ matrix with ghost entries:

$$\mathcal{B}^{(l)}_{(m)}(z,\overline{z}) \equiv \sum_{n=0}^{(s-1)} \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} \partial_{(p)}\mathcal{K}^{(n)}(z,\overline{z})\mathcal{R}^{(l)}_{(m+n-p)}(z,\overline{z}) \tag{3.16}$$

The transformations in eq (3.15) give rise to a $\mathcal{W}(s-1)$ algebra [21] ¿From this is easy to realize, for $1 \leq i \leq s-1$:

$$\delta_{\mathcal{W}}\varpi^{J}_{i}(z,\overline{z}) = \sum_{n} \mathcal{B}^{(n)}_{(i)}(z,\overline{z})\varpi^{(J)}_{(n)}(z,\overline{z}) \tag{3.17}$$

$$\delta_{\mathcal{W}}\mathcal{B}(z,\overline{z}) = \mathcal{B}(z,\overline{z})\mathcal{B}(z,\overline{z}) \tag{3.18}$$

The previous equation (3.18) awards to the composite fields $\mathcal{B}(z,\overline{z})$ a relevance, since, using this parametrization, a $GL(s-1)$ structure comes into evidence. This symmetry is not evident using the $\mathcal{K}(z,\overline{z})$ ghosts reparametrization.

The reduction coefficients $\mathcal{R}^{(l)}_{(m)}(z,\overline{z})$ allow the closure of the related algebra,by means the property endowed in Eq(2.5). This has completely distorted the regular procedure; and, more, in order to provide the Jacoby identity the $\mathcal{R}^{(l)}_{(m)}(z,\overline{z})$ have to transform under the B.R.S. algebra as:

$$\delta_{\mathcal{W}}\mathcal{R}^{(p)}_{(n)}(z,\overline{z}) = \mathcal{B}^{(p)}_{(n)}(z,\overline{z}) - \sum_{q} \mathcal{R}^{(q)}_{(n)}(z,\overline{z})\mathcal{B}^{(p)}_{(q)}(z,\overline{z}) \tag{3.19}$$

It follows from Eq (3.18):

$$\delta_{\mathcal{W}}Tr\mathcal{B}(z,\overline{z}) = \delta_{\mathcal{W}}\det\mathcal{B}(z,\overline{z}) = 0 \tag{3.20}$$

and from eq (2.8)[7]

$$f^{(1)}(z,\overline{z}) = \det\varpi(z,\overline{z})^{-\frac{1}{s}} \tag{3.21}$$

It is important to stress, as widely investigated in reference [21], that the ghosts $\mathcal{K}^{(l)}(z,\overline{z})$ are not tensors, since under finite holomorphic rescaling they transform as:

$$\mathcal{K}^{(m)}(w,\overline{w}) = \mathcal{K}^{(l)}(z,\overline{z})\Phi^{-1}{}^{m}_{(\ell)}(z) \tag{3.22}$$

$$\varpi^{(R)}_{(m)}(w,\overline{w}) = \Phi^{(\ell)}_{m}(z)\varpi^{(R)}_{(\ell)}(z,\overline{z}) \tag{3.23}$$

where:

$$\Phi^{-1}_{\ell,k}(z) = \begin{cases} w^{(\ell)}(z)\,\delta_{k,1}\ , \\ \\ \displaystyle\sum_{r=\ell-1}^{k-1} \frac{(k-1)!\,w^{(k-r)}(z)}{(k-r-1)!} \sum_{\substack{a_1+\cdots+ra_r=r \\ a_1+\cdots+a_r=\ell-1}} \left(\prod_{n=1}^{r} \frac{1}{a_n!}\left(\frac{w^{(n)}(z)}{n!}\right)^{a_n}\right) \\ ,\ell \geq k \geq 2 \\ \\ 0, \ell < k \end{cases} \tag{3.24}$$

with non vanishing determinant, $\det \Phi^{-1}(z) = (w'(z))^{s(s-1)/2}$.

Under holomorphic reparametrization the ghosts $\mathcal{B}^{(l)}_{(j)}(z,\overline{z})$ behave as:

$$\mathcal{B}^{(l)}_{(j)}(w,\overline{w}) = \sum_{(s,m)} \Phi^{(s)}_{(j)}(z)\mathcal{B}^{(m)}_{(s)}(z,\overline{z})\Phi^{(-1)}{}^{(l)}_{(m)}(z) \tag{3.25}$$

A reduction from jets to tensor can be found in ref [21], and it is crucial from the physical point of view, since, for instance, tensor objects are suitable for constructing globally defined observables.

Let $\mathcal{C}^{(l)}(z,\overline{z})$ be holomorphic tensor ghosts generalizing[16] the Becchi's ones[28] , the decomposition:

$$\mathcal{K}^{(l)}(z,\overline{z}) = \mathcal{C}^{(l)}(z,\overline{z}) + \sum_{p>l;0\leq r\leq p-l} \partial_r \mathcal{C}^{(p)}(z,\overline{z})\mathcal{T}^{(r,l)}_{(p)}(z,\overline{z}) \tag{3.26}$$

where:

$$
\begin{aligned}
&\mathcal{T}^{(b,m)}_{(p)}(z,\bar{z}) = 0 \quad for \quad p < m \\
&\mathcal{T}^{(r,m)}_{(p)}(z,\bar{z}) = \delta^{(r)}_{(0)} \quad for \quad p = m
\end{aligned}
\tag{3.27}
$$

satisfy all the consistency conditions. By the way the holomorphic ghosts can be decomposed into the non-holomorphic ones $c^{(r,s)}(z,\bar{z})$[17]:

$$
\mathcal{C}^j(z,\bar{z}) = \sum_{r,s=1\cdots j} \left[r!s! \left(\Pi_i \frac{\left(\mu^{l_i}_{(\bar{z})}(z,\bar{z})\right)^{k_i}}{k_i!} \right) |_{\left\{ \begin{array}{c} \sum_i k_i = s \\ r + \sum_i l_i k_i = j \end{array} \right\}} \right] c^{(r,s)}(z,\bar{z}) \tag{3.28}
$$

where $\mu^{l_i}_{(\bar{z})}(z,\bar{z})$ are the Bilal-Fock-Kogan multipliers [29] [16][30][21]. To shorten we compress:

$$
c^{(1,0)}(z,\bar{z}) \equiv c(z,\bar{z}) \tag{3.29}
$$

$$
c^{(0,1)}(z,\bar{z}) \equiv \bar{c}(z,\bar{z}) \tag{3.30}
$$

3.1 B.R.S algebra and D.O.R. decomposition

Our algebra describes a spatial transformation, and must include (in its lower order derivative term) the infinitesimal factorized diffeomorphism in terms of the holomorphic ghosts (3.28). In this approximation the derivative operator can be described in an algebraic way, using the Fock space formalism [31] [32], using each field and its derivatives are considered as independent variables.

If one writes the B.R.S. operator over the Fock space it can be shown [33][34] that:

$$
\partial = \left\{ \frac{\partial}{\partial c(z,\bar{z})}, \delta_{\mathcal{W}} \right\} \tag{3.31}
$$

$$
\bar{\partial} = \left\{ \frac{\partial}{\partial \bar{c}(z,\bar{z})}, \delta_{\mathcal{W}} \right\} \tag{3.32}
$$

Statement 3.1 *The closure of the algebra (i.e the nilpotency of the B.R.S operator) and the links Eqs (3.31),(3.32): allow to get rid of the framework which led to eq (2.1), (2.3), and fully contains (2.5).* `It is thus possible to encapsulate the Laguerre-Forsyth formalism in a B.R.S. way`

¿From (3.17) we get:

$$\partial \varpi^{(R)}_{(q)}(z,\overline{z}) - \sum_r \frac{\partial \mathcal{B}^{(r)}_{(q)}(z,\overline{z})}{\partial c(z,\overline{z})} \varpi^{(R)}_{(r)}(z,\overline{z}) = 0 \tag{3.33}$$

$$\overline{\partial} \varpi^{(R)}_{(q)}(z,\overline{z}) - \sum_r \frac{\partial \mathcal{B}^{(r)}_{(q)}(z,\overline{z})}{\partial \overline{c}(z,\overline{z})} \varpi^{(R)}_{(r)}(z,\overline{z}) = 0 \tag{3.34}$$

Where from Eqs (3.27) (3.28) and eq (3.16) we can derive :

$$\frac{\partial \mathcal{B}^{(m)}_{(n)}(z,\overline{z})}{\partial c(z,\overline{z})} = \mathcal{R}^{(m)}_{(n+1)}(z,\overline{z}) \equiv \mathcal{J}_{(z)}{}^{(m)}_{(n)}(z,\overline{z}) \tag{3.35}$$

in details:

$$\mathcal{J}_{(z)}(z,\overline{z}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \mathcal{R}^{(1)}_{(s)}(z,\overline{z}) & \mathcal{R}^{(2)}_{(s)}(z,\overline{z}) & \cdots & \cdots & \cdots & \mathcal{R}^{(s-1)}_{(s)}(z,\overline{z}) \end{pmatrix}. \tag{3.36}$$

which gives (in the equation (3.33)) the B.R.S. origin of the Eq (2.5), and then imply the D.O.R. mechanism.

Moreover we have a further link:

$$\begin{aligned} \frac{\partial \mathcal{B}^{(k)}_{(l)}(z,\overline{z})}{\partial \overline{c}(z,\overline{z})} &= \sum \begin{pmatrix} l \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \partial_{(j+r)} \mu^{(p)}(z,\overline{z}) \partial_{(i-j)} \mathcal{T}^{(r,m)}_{(p)}(z,\overline{z}) \mathcal{R}^{(k)}_{(l-i+m)}(z,\overline{z}) \\ &\equiv \mathcal{J}_{(\overline{z})}{}^{(k)}_{(l)}(z,\overline{z}) \end{aligned} \tag{3.37}$$

which appears as something new with respect the D.O.R. expansion.

So if we introduce the connection 1-form:

$$\mathbf{J}(z,\overline{z}) = \mathcal{J}_{(z)}(z,\overline{z}) dz + \mathcal{J}_{(\overline{z})}(z,\overline{z}) d\overline{z} \tag{3.38}$$

The components of the connection in Eqs (3.35) (3.37) verify, as we can see by a direct application of the commutation rules Eqs (3.31) and (3.32) on Eq (3.18),

$$\delta_{\mathcal{W}}\mathbf{J}(z,\overline{z}) = -\mathbf{d}\mathcal{B}(z,\overline{z}) + \Big[\mathcal{B}(z,\overline{z}), \mathbf{J}(z,\overline{z})\Big]_{\text{grad}} \tag{3.39}$$

(where $\mathbf{d} = dz\partial + d\overline{z}\overline{\partial}$ is the differential)

Further Eqs (3.33)(3.34) sum up into:

$$\mathcal{D}_{\mathbf{J}}\varpi^{(R)}_{(q)}(z,\overline{z}) = 0 \tag{3.40}$$

namely, $\mathbf{d}\varpi - \mathbf{J}\varpi = 0 \Rightarrow \mathbf{d}(\mathbf{J}\varpi) = 0$.

Moreover the use of Eqs (3.31)(3.32) on (3.39) provide the zero curvature condition:

$$\mathbf{dJ}(z,\overline{z}) - \frac{1}{2}\Big[\mathbf{J}(z,\overline{z}), \mathbf{J}(z,\overline{z})\Big]_{\text{grad}} = 0 \tag{3.41}$$

which implies ${\mathcal{D}_{\mathbf{J}}}^2 = 0$.

This condition, applied to the decomposition Eq (3.26) fixes $\mathcal{T}^{(b,m)}_{(p)}(z,\overline{z})$ as functions of $\mathcal{R}^{(k)}_{(l)}(z,\overline{z})$ such that:

$$\sum_{i=0}^{l}\binom{l}{i}\sum_{j=0}^{i}\binom{i}{j}\Big[\partial_{(i-j)}\mathcal{T}^{((b-j)-1,m)}_{(p)}(z,\overline{z}) + \partial_{(i-j+1)}\mathcal{T}^{(b-j,m)}_{(p)}(z,\overline{z})\Big]\mathcal{R}^{(k)}_{(l-i+m)}(z,\overline{z}) = 0$$
$$p - m \geq b \geq 0; p, m = 1\cdots s-1; \tag{3.42}$$

We can verify that equation Eqs (3.34) takes its consistency directly from the zero curvature condition,as we can see by applying a $\overline{\partial}$ operator to the first one and using the relations, derived from eq (3.19):

$$\overline{\partial}\mathcal{J}_{(z)}{}^{(q)}_{(n)}(z,\overline{z}) + \sum_{l}\mathcal{J}_{(z)}{}^{(q)}_{(l)}\mathcal{J}_{(\overline{z})}{}^{(l)}_{(n)} = \mathcal{J}_{(\overline{z})}{}^{(q)}_{(n+1)}(z,\overline{z}) \tag{3.43}$$

$$\partial\mathcal{J}_{(z)}{}^{(q)}_{(n)}(z,\overline{z}) + \sum_{l}\mathcal{J}_{(z)}{}^{(q)}_{(l)}\mathcal{J}_{(z)}{}^{(l)}_{(n)} = \mathcal{J}_{(z)}{}^{(q)}_{(n+1)}(z,\overline{z}) \tag{3.44}$$

All the previous equations can be viewed as holonomy constraints, describing the independent parallel transport along the z and $\overline{z}$ axes of the wronskian $\varpi^{(R)}_{(q)}(z,\overline{z})$

by means of a zero curvature connection $\mathbf{J}$, and hint that $\varpi^{(R)}_{(q)}(z,\overline{z})$ (and then $Z^{(R)}(z,\overline{z})$) may be, in general, a non-local function on $\mathcal{J}_{(z)}, \mathcal{J}_{(\overline{z})}$ (but local in $(z,\overline{z})$!)

We emphasize once more that we have brought the analysis on a B.R.S. algebraic framework, from a classical branch of Riemaniann geometry. This implies that we have to assure that our treatment will lead to an unique and reliable solution. The first requirement is that this solution has to be stable under the algebra deformation; we will now show in the next, that some problem is at the door.

3.2 The orbits of the gauge fields: the non uniqueness of the coordinates definition

Let us introduce the fields $\Lambda^{(m)}_{(n)}(z,\overline{z})$ transforming under $\mathcal{W}$, according to the adjoint representation with the $\mathcal{B}^{(m)}_{(n)}(z,\overline{z})$ ghosts:

$$\delta_{\mathcal{W}}\Lambda^{(m)}_{(n)}(z,\overline{z}) = \sum_{r=1}^{s-1} \mathcal{B}^{(r)}_{(n)}(z,\overline{z})\Lambda^{(m)}_{(r)}(z,\overline{z}) - \mathcal{B}^{(m)}_{(r)}(z,\overline{z})\Lambda^{(r)}_{(n)}(z,\overline{z}) \tag{3.45}$$

and the fields $\Lambda^{(m)}_{(n)}(z,\overline{z})$ rescale under holomorphic change of charts as:

$$\Lambda^{(m)}_{(n)}(w,\overline{w}) = \sum_{r,p} \Phi^{(m)}_{(r)}(z)\Lambda^{(r)}_{(p)}(z,\overline{z})(\Phi)^{-1}{}^{(p)}_{(n)}(z) \tag{3.46}$$

Now if we introduce:

$$\varpi'^{(R)}_{(n)}(z,\overline{z}) = \Lambda^{(m)}_{(n)}(z,\overline{z})\varpi^{(R)}_{(m)}(z,\overline{z}) \tag{3.47}$$

this quantities have the B.R.S. variations:

$$\delta_{\mathcal{W}}\varpi'^{(J)}_{(n)}(z,\overline{z}) = \sum_{n} \mathcal{B}^{(m)}_{(n)}(z,\overline{z})\varpi'^{(J)}_{(m)}(z,\overline{z}) \tag{3.48}$$

So eq (3.31) and (3.32) give:

$$\mathcal{D}_{(\mathbf{J})}\varpi'^{(R)}_{(q)}(z,\overline{z}) = 0 \tag{3.49}$$

So,if we define the coordinates $Z'^{(R)}(z,\overline{z})$ by means of the differential equations:

$$\partial_{(n)}Z'^{(R)}(z,\overline{z}) \equiv \varpi'^{(R)}_{(n)} = \left(\Lambda^{(m)}_{(n)}(z,\overline{z})\partial_{(m)}Z^{(R)}(z,\overline{z})\right) \tag{3.50}$$

(so they are, in general, non local in Λ!) it is straightforward to get:

$$\begin{aligned}
\partial\left(\partial_{(n)}Z'^{(R)}(z,\overline{z})\right) &= \mathcal{J}_{(z)}{}^{(r)}_{(n)}(z,\overline{z})\left(\partial_{(m)}Z'^{(R)}(z,\overline{z})\right)\\
\overline{\partial}\left(\partial_{(n)}Z'^{(R)}(z,\overline{z})\right) &= \mathcal{J}_{(\overline{z})}{}^{(r)}_{(n)}(z,\overline{z})\left(\partial_{(m)}Z'^{(R)}(z,\overline{z})\right)
\end{aligned} \tag{3.51}$$

So the coordinates $Z'^{(R)}(z,\overline{z})$ satisfy the same B.R.S. algebra (3.17) and have the identical D.O.R. expansion (2.5) as the $Z^{(R)}(z,\overline{z})$ ones.

One may wonder what is the spatial meaning of the additional symmetry represented in the Eq (3.45). By applying (3.31) and (3.32) on the latter, the Λ's have to verify the conditions:

$$\partial\Lambda^{(m)}_{(n)}(z,\overline{z}) - (\mathcal{J}_{(z)})^{(r)}_{(n)}(z,\overline{z})\Lambda^{(m)}_{(r)}(z,\overline{z}) + (\mathcal{J}_{(z)})^{(m)}_{(r)}(z,\overline{z})\Lambda^{(r)}_{(n)}(z,\overline{z}) = 0 \tag{3.52}$$

$$\overline{\partial}\Lambda^{(m)}_{(n)}(z,\overline{z}) - (\mathcal{J}_{(\overline{z})})^{(r)}_{(n)}(z,\overline{z})\Lambda^{(m)}_{(r)}(z,\overline{z}) + (\mathcal{J}_{(\overline{z})})^{(m)}_{(r)}(z,\overline{z})\Lambda^{(r)}_{(n)}(z,\overline{z}) = 0 \tag{3.53}$$

or in a more compact notation $\mathbf{d}\Lambda - [\mathbf{J},\Lambda] = 0$, i.e.

$$\left(\mathcal{D}_{(\mathbf{J})}\Lambda\right)^{(m)}_{(n)}(z,\overline{z}) = 0 \tag{3.54}$$

which links the derivatives of Λ to the $\mathbf{J}$ connection.

It is evident to realize that this is not the more general solution we can get: in fact if we define $\Lambda'^{(m)}_{(n)}(z,\overline{z}) = (\Lambda^r)^{(m)}_{(n)}(z,\overline{z})$ for all r integers, (which means that $(\Lambda^r)^{(m)}_{(n)}(z,\overline{z})$ is an invertible matrix in $GL(s-1,\mathbf{C})$) we can get the same conclusions.

We may ask now which principle selects the best solution among these various possibilities. We believe that this might only come from a dynamical foreground, since the geometry alone does not provide the solution. For this reason we introduce a $\mathcal{W}(s)$ invariant Lagrangian model, with equation of motion the equations (3.40)and (3.54), in order to investigate the dynamical properties of the Forsyth-Laguerre coordinates $Z^{(R)}(z,\overline{z})$ both at the classical and quantum levels.

First of all, we have to find the more general Lagrangian constrained by the $\mathcal{W}$ symmetry and giving rise to the all the algebraic requirements for consistency.

But first we have to remark the fundamental difference between Eqs (3.40) (3.54): while the first ones (as previously widely underlined) assures the closure of the algebra and the nilpotency of the B.R.S. operator, the last ones give only consistency condition for the $\Lambda^{(m)}_{(n)}(z,\overline{z})$ fields, due the space transformation character of the $\mathcal{W}$ symmetry.

4 Lagrangian Field Theory model

4.1 The classical model

We want here build a polynomial Lagrangian invariant under $\mathcal{W}(s)$ containing $\varpi(z,\overline{z})$, $\mathcal{J}_{(z)}(z,\overline{z})$, $\mathcal{J}_{(\overline{z})}(z,\overline{z})$ and $\Lambda(z,\overline{z})$, as independent fields.

A natural further need is (obviously) the invariance under holomorphic change of charts for the Action integrand. This requirement becomes outstanding, owing to the unusual (jets) behaviour of these fields.

The aim of this dynamical model is to resolve the degeneracy of the non complete fixing of the coordinates $Z^{(r)}(z,\overline{z})$ with the same D.O.R. decomposition.

Obviously this is not expected at the classical level, but we foresee that quantum fluctuations could disentangle the problem.

The classical generating functional is defined as:

$$\mathcal{Z}_{Classical}[\mathbb{J}] = \int \prod \left[dZ^{(R)} d\mathcal{J}_{(z)} d\mathcal{J}_{(\overline{z})} d\Lambda \right] \exp^{\frac{i}{\hbar}\Gamma_{Classical}} \tag{4.55}$$

where $\mathbb{J}$ labels collectively all the external classical fields on which the functional integration is not performed. So we introduce the Classical Action which has to be invariant under both holomorphic change of charts and the B.R.S. algebra :

$$\begin{aligned}
\Gamma_{Classical} = \int dz \wedge d\bar{z} Tr \Bigg[\sum_{n=0}^{\infty} \Bigg(\eta_n \Lambda(z,\overline{z})^n Tr(\Lambda^{n'}(z,\overline{z})) \det(\Lambda^{n"}(z,\overline{z})) \\
\mathcal{D}_{(z)}\Lambda(z,\overline{z})\mathcal{D}_{(\overline{z})}\Lambda(z,\overline{z}) + \sum_{R,S} \alpha_{(z\bar{z},n,p)}(z,\overline{z})\Lambda^n(z,\overline{z}) Det\Lambda^p(z,\overline{z}) Tr(\Lambda^q(z,\overline{z})) \Bigg) \\
+\beta_{(\overline{z})}(z,\overline{z}) \left[\mathcal{J}_{(z)}(z,\overline{z}), \Lambda(z,\overline{z}) \right] + \rho(z,\overline{z}) \left[\mathcal{J}_{(\overline{z})}(z,\overline{z}), \mathcal{J}_{(z)}(z,\overline{z}) \right] \\
+\beta_{(z)}(z,\overline{z}) \left[\mathcal{J}_{(\overline{z})}(z,\overline{z}), \Lambda(z,\overline{z}) \right] + J_{z\bar{z}(\Lambda)}(z,\overline{z})\Lambda(z,\overline{z})
\end{aligned}$$

$$
\begin{aligned}
&+J_{z\bar{z}(R)}(z,\overline{z})Z^{(R)}(z,\overline{z})+\eta_{(R)}{}^{(n)}_{(\overline{z})}(z,\overline{z})\mathcal{J}^{(p)}_{(z),(n)}\varpi^{(R)}_{(p)}(z,\overline{z})\\
&+\eta_{(R)}{}^{(n)}_{(z)}(z,\overline{z})\mathcal{J}^{(p)}_{(\overline{z}),(n)}\varpi^{(R)}_{(p)}(z,\overline{z})\gamma_{z\bar{z}(R)}(z,\overline{z})\delta_{\mathcal{W}}Z^{(R)}(z,\overline{z})+\sigma_{(z)}(z,\overline{z})\mathcal{J}_{(\overline{z})}(z,\overline{z})\\
&+\sigma_{(\overline{z})}(z,\overline{z})\mathcal{J}_{(z)}(z,\overline{z})+\gamma_{\mathcal{J}_{(z)}}(z,\overline{z})\delta_{\mathcal{W}}\mathcal{J}_{(z)}(z,\overline{z})+\gamma_{\mathcal{J}_{(\overline{z})}}(z,\overline{z})\delta_{\mathcal{W}}\mathcal{J}_{(\overline{z})}(z,\overline{z})\\
&\left.+\gamma_{\Lambda}(z,\overline{z})\delta_{\mathcal{W}}\Lambda(z,\overline{z})+\gamma_{\mathcal{B}}(z,\overline{z})\mathcal{B}(z,\overline{z})+\zeta(z,\overline{z})\delta_{\mathcal{W}}\mathcal{B}(z,\overline{z})\right]
\end{aligned}
\tag{4.56}
$$

The Action is endowed with all the Classical external fields necessary for a proper tratment in the functional approach. The canonical dimensions of the fields are fixed by the "little" indices content and their $\Phi\Pi$ charge[33]:

$$
dim = N_{down} - N_{up} + Q_{\Phi\Pi} \tag{4.57}
$$

The algebra closes if both Eqs (3.40) and (3.41) hold; furthermore we have to fix eq (3.54) as consistency conditions.

$$
\partial_{(n+1)}\frac{\delta}{\delta J_{(z\bar{z})}{}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}=\frac{\delta}{\delta\eta_{(\overline{z})}{}^{(n)}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.58}
$$

$$
\overline{\partial}\partial_{(n)}\frac{\delta}{\delta J_{(z\bar{z})}{}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}=\frac{\delta}{\delta\eta_{(z)}{}^{(n)}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.59}
$$

These equations are nothing else that the Eqs (3.40) written in a functional language.In the context of a Lagrangian Fields Theory model, they represent the equations of motions of the fields $Z^{(R)}(z,\overline{z})$, in order to assure the large diff algebra closure in the (z) direction (eq (4.58)) and in the $(\overline{z})$ one (eq (4.59)) with a parallel transport carried by a zero curvature connection.

To the previous equations, must be joined the consistency conditions:

$$
\overline{\partial}\frac{\delta}{\delta\eta_{(\overline{z})}{}^{(n)}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}=\partial\frac{\delta}{\delta\eta_{(z)}{}^{(n)}_{(R)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.60}
$$

$$
\overline{\partial}\frac{\delta}{\delta J_{z\bar{z}\Lambda}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}=\frac{\delta}{\delta\beta_{z}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.61}
$$

$$
\partial\frac{\delta}{\delta J_{z\bar{z}\Lambda}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}=\frac{\delta}{\delta\beta_{(\overline{z})}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.62}
$$

$$\overline{\partial}\frac{\delta}{\delta\sigma_{(\overline{z})}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} - \partial\frac{\delta}{\delta\sigma_{(z)}(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]}$$
$$= \frac{\delta}{\delta\rho(z,\overline{z})}\mathcal{Z}_{Classical}[\mathbb{J}]|_{[\mathbb{J}=0]} \tag{4.63}$$

which respectively reproduce the compatibility condition for (3.40), Eqs (3.52) and (3.53), and the vanishing of the curvature (3.41).

Finally the $\alpha_{(z\bar{z},k,n,R,S)}{}^{(m)}_{(n)}(z,\overline{z})$ fields in Eq (4.56) have been introduced to get the right change of chart property of their relative terms, and behave under the $\mathcal{W}$ adjoint representation:

$$\delta_{\mathcal{W}}\alpha_{(z\bar{z},k,n)}{}^{(m)}_{(n)}(z,\overline{z})$$
$$= \sum_r \Big(\mathcal{B}^{(r)}_{(n)}(z,\overline{z})\alpha_{(z\bar{z},k,n)}{}^{(m)}_{(r)}(z,\overline{z}) - \mathcal{B}^{(m)}_{(r)}(z,\overline{z})\alpha_{(z\bar{z},k,n)}{}^{(r)}_{(n)}(z,\overline{z}) \Big) \tag{4.64}$$

We remark again that the algebra is nilpotent and consistent if the equations of motion eqs(4.58)(4.61)(4.59)(4.62)(4.63) hold as consistency conditions.

It is evident from the Action Eq (4.56) that the $\Lambda(z,\overline{z})$ fields have a non trivial two point function:

$$< \Lambda\Lambda_{(p)} >^{(-1)m,n}_{\quad p,q} = p^2\delta^{(i)}_{(j)}\delta^{(m)}_{(n)} + \sum_R < \widetilde{\alpha}, 2, 0 >^{(i,m)}_{(j,n)}$$
$$+ < \widetilde{\alpha}, 0, 1 >_{\mathcal{W}(3)}{}^{(i,m)}_{(j,n)} + < \widetilde{\alpha}, 0, 2 >_{\mathcal{W}(2)}{}^{(i,m)}_{(j,n)} \tag{4.65}$$

where the last two terms come from the Λ determinants in the case of $s = 3, 2$ respectively. Its inverse produces a non trivial propagator: We shall indicate the $< \Lambda\Lambda_{(p)} >^{m,n}_{p,q}$ propagator with a wavy line as:

$$< \Lambda\Lambda_{(p)} >^{m,n}_{p,q} \tag{4.66}$$

On the other hand the fields $\mathcal{J}_{(z)}(z,\overline{z})$, $\mathcal{J}_{(\overline{z})}(z,\overline{z})$ and the coordinates $Z^{(R)}(z,\overline{z})$ have no propagator at the classical level.

The quantum improvement of the model wants to extend at each order of the $\hbar$ loop expansion the $\mathcal{W}$ symmetry; that is it demands to find an extension af the Classical Action Γ'_0. If not so the breakings Δ:

$$\delta_{\mathcal{W}}\Gamma'_0 = \Delta \tag{4.67}$$

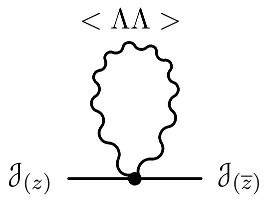
have to satisfy, due to the Wess-Zumino consistency condition:

$$\delta_{\mathcal{W}}\Delta_{\mathcal{W}} = 0 \tag{4.68}$$

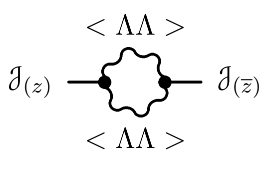
A complete solution for this problem can be found in the literature of the B.R.S. approach to Gauge Fields models[35] . Indeed it easy (but troublesome and tedious) to see, using the consistency conditions coming from eq (4.68), that the external fields component of the anomaly can be compensated within a counter-term procedure. So only the quantized field component survives. It has been noted [36] that introducing cocycle terms and total derivatives,the anomaly can be written in a well defined form as:

$$\begin{aligned} \Delta_{\mathcal{W}_2}^1(z,\bar{z}) &= 3\sigma Tr(\mathcal{J}_{(\bar{z})}(z,\bar{z})\partial\mathcal{B}(z,\bar{z}) - \mathcal{B}(z,\bar{z})\partial\mathcal{J}_{(\bar{z})}(z,\bar{z}) \\ &+ 2\mathcal{B}(z,\bar{z})(\mathcal{J}_{(z)}(z,\bar{z})\mathcal{J}_{(\bar{z})}(z,\bar{z}) - \mathcal{J}_{(\bar{z})}(z,\bar{z})\mathcal{J}_{(z)}(z,\bar{z})))dz\wedge d\bar{z} \end{aligned} \tag{4.69}$$

In a descriptive point of view the anomaly originates from the arising, at the quantum level, by blowing up the $\mathcal{J}_{(z)}(z,\bar{z})\mathcal{J}_{(\bar{z})}(z,\bar{z})\Lambda(z,\bar{z})$ vertices by means of Λ propagators, in order to generate as 1-loop contributions, tyhe $< \mathcal{J}_{(z)}(z,\bar{z})\mathcal{J}_{(\bar{z})}(z',\bar{z}') >$ two point functions. This construction generates divergencies which require counterterm subtractions from the diagram:

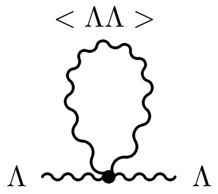


(4.70)



(4.71)

We point out that the $\Lambda\Lambda$ loops are the only pathologies which introduce divergencies (and so needs a subtraction procedure). These phenomena occur also in the pure Λ sector: for instance in the 1-loop correction to the Λ propagator:



(4.72)

(and similar diagram to that in fig. (4.71)) and more generally in the n-point correlation function for the Λ fields, but in this case the counter-term fit to remove this pathology, is already present at the Classical level.

In order to cancel the anomaly (4.69) and the divergencies coming from the $< \mathcal{J}_{(z)}(z,\overline{z})\mathcal{J}_{(\overline{z})}(z',\overline{z}') >$ two-point function we follow a method alternative to the one already given in Ref [19], which reminds to the most classical papers in gauge theory renormalization [37] [38] [36]

We shall introduce as counter-term the three dimensional Chern-Simons action:

$$\Gamma_{counterterm} = \hbar\sigma \int dz \wedge d\bar{z} \wedge dt \Bigg\{ Tr \Bigg(\mathbf{J}(z,\overline{z},t) d\mathbf{J}(z,\overline{z},t) + \frac{2}{3}\mathbf{J}(z,\overline{z},t)\mathbf{J}(z,\overline{z},t)\mathbf{J}(z,\overline{z},t) \Bigg) \Bigg\} \tag{4.73}$$

to define a new Action:

$$\Gamma_{Quantum} = \Gamma_{Classical} - \Gamma_{counterterm} \tag{4.74}$$

The σ coefficient can be calculated in a well defined renormalization scheme, and its value fixes the coupling of the pointlike **JJ**, **JJJ** interactions.

So the model is now anomaly free, and a new generating functional $\mathcal{Z}_{Quantum}[\mathbb{J}]$, which is $\mathcal{W}$ invariant at the quantum level, can be defined, so extending at every order of perturbation theory the Equation (4.55).

$$\mathcal{Z}_{Quantum}[\mathbb{J}] = \int \prod \left[dZ^{(R)} d\mathcal{J}_{(z)} d\mathcal{J}_{(\overline{z})} d\Lambda \right] \exp^{\frac{i}{\hbar}\Gamma_{Quantum}} \tag{4.75}$$

$$\delta_{\mathcal{W}}\mathcal{Z}_{Quantum}[\mathbb{J}] = 0 \tag{4.76}$$

The price to pay is to introduce 1-loop contributions generating the two point function $< \mathbf{J}(z,\overline{z})\mathbf{J}(z,\overline{z}) >$ (which, as counter-term, compensate the graphs (4.70)(4.71)) and the 3-point function $< \mathbf{J}(z,\overline{z})\mathbf{J}(z,\overline{z})\mathbf{J}(z,\overline{z}) >$.

This leads to a new dynamics at the quantum level, where the $Z^{(R)}(z,\overline{z})$ fields still do not propagate.

But we never omit that the symmetry is true and consistent only with the addition, of Eqs(4.58), (4.61) (4.59) (4.62), (4.63), with $\mathcal{Z}_{Quantum}$ replacing $\mathcal{Z}_{Classical}$.

So the quantum extension, linked to the conclusions found, at the Classical level after the discovery of the D.O.R equivalence of the coordinates $Z^{(R)}(z,\overline{z})$ and $Z'^{(R)}(z,\overline{z})$ in Eqs (3.51), generates new prospects for the meaning of Forsyth-Laguerre coordinates and their frames.

4.3 The dynamics of Forsyth Laguerre frames

The first requirement for a coordinate system, is the "absolute" independence in the choice of each coordinate.

This means that each $Z^{(R)}(z,\overline{z})$ field has to be "completely independent" from the others, not only in the "mathematical" sense but even in the "dynamical " one.

Recall that at the Classical level there is no interaction for these fields, but due to the birth, at the quantum level, of the two point function $< \mathcal{J}_z\mathcal{J}_{(\overline{z})} >$, from the added counter-term, this new propagator represented as:

$$< \mathcal{J}_{z}{}^{(p)}_{(m)}(z,\overline{z})\mathcal{J}_{\bar{w}}{}^{(q)}_{(n)}(w,\overline{w}) > \tag{4.77}$$

induces a new dynamical mechanism, and is at the origin of the interaction between coordinates $Z^{(R)}(z,\overline{z})$, $Z^{(S)}(w,\overline{w})$ of two different Forsyth-Laguerre frames, which at the Classical level were totally independent.

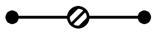
Geometrically this might be the signature of the existence of a propagator linking two different fibers over the Riemann surface.

Indeed, if we consider the correlator of two first order derivatives of the wronskian, the D.O.R. decomposition produces an interaction of each wronkian element with the the connection $\mathbf{J}(z,\overline{z})$, that is: ($\mathbb{R}$ means radial ordering)

$$\langle \mathbb{R}\partial\varpi^{(R)}_{(p)}(z,\overline{z})\overline{\partial}_{\overline{w}}\varpi^{(S)}_{(q)}(w,\overline{w})\rangle = \partial_z^{(p+1)}\partial_w^{(q)}\overline{\partial}_{\overline{w}}\langle Z^{(R)}(z,\overline{z})Z^{(S)}(w,\overline{w})\rangle$$

$$= \mathbb{R}\left(\partial_{z\,(p+1)}\frac{\delta}{\delta J_{(z\bar{z})}{}_{(R)}(z,\overline{z})}\overline{\partial}_{\overline{w}}\partial_{w\,(q)}\frac{\delta}{\delta J_{(w\bar{w})}{}_{(S)}(w,\overline{w})}\mathcal{Z}^{Connected}_{Quantum}[\mathbb{J}]|_{[\mathbb{J}=0]}\right)$$

$$= \mathbb{R}\left(\frac{\delta}{\delta\eta_{(\overline{z})}{}^{(p)}_{(R)}(z,\overline{z})}\frac{\delta}{\delta\eta_{(w)}{}^{(q)}_{(S)}(w,\overline{w})}\mathcal{Z}^{Connected}_{Quantum}[\mathbb{J}]|_{[\mathbb{J}=0]}\right) =$$

$\sum_{m,n}\varpi^{(R)}_{(m)}(z,\overline{z})$  $\varpi^{(S)}_{(n)}(w,\overline{w})$

$$< \mathcal{J}_z{}^{(p)}_{(m)}(z,\overline{z})\mathcal{J}_{\bar{w}}{}^{(q)}_{(n)}(w,\overline{w}) >$$

$$\equiv \hbar\theta^{(R,S)}_{(p+q+1,1)}(z-w,\overline{z}-\overline{w}) \tag{4.78}$$

where the last line is to underline that the lowest order (local) term comes from the counter-term addition, so it takes a non vanishing value only at the quantum extension.

So we have to ask if the Forsyth-Laguerre coordinate frames loose their meaning (in the sense of their full independence) after the quantum corrections or not.

To get a precise answer to this question we have to make use of the locality attribute of the frame in Eq (2.4); that is that `at each point of the Riemann surface` lives a frame, which, at the Classical level, is fully disjoint to the other ones living on the other points of the manifold.

On the other hand the radiative corrections generate an $\hbar$-order interaction of the derivatives of the wronskians rows $\varpi^{(R)}(z,\overline{z})$ and $\varpi^{(S)}(w,\overline{w})$, and induce their interaction depending from the distance $(z-w,\overline{z}-\overline{w})$. Indeed the counter-term addition restores the $\mathcal{W}$-type extended holomorphic diffeomorphism invariance (and then the conformal one.

So, from a pure mathematical point of view, the quantum dynamics puts restrictions to the "full" independence of the coordinates $Z^{(R)}(z,\overline{z})$ from each others, and the definition of the Laguerre-Forsyth frame, at the quantum level, must be reconsidered.

Indeed the $Z^{(R)}(z,\overline{z})$ space-time manifold is now replaced by a quantum Hilbert space and its use as a coordinates underlying space is really troublesome.

The only escape way to use, is to recover the role of the Laguerre-Forsyth frame, after the quantum improvement, introducing, by means of the dynamics rules, a "quantum star" [3] product within the projective Laguerre-Forsyth coordinates frame

[3] in the sense that it is meaningless at the Classical level

at point $(z-w, \overline{z}-\overline{w})$ as:

$$\langle Z^{(R)} * Z^{(S)} \rangle (z-w, \overline{z}-\overline{w}) \equiv \langle Z^{(R)}(z, \overline{z}) Z^{(S)}(w, \overline{w}) \rangle = \hbar\theta^{(R,S)}(z-w, \overline{z}-\overline{w}) \tag{4.79}$$

We stress that this "composition rule" is dynamically induced by a Quantum Conformal Field Theory model in which the extended holomorphic diffeomorphism symmetry has been restored after the addition of a Chern-Simons counter-term.

This conclusion calls to mind the papers [39],[40],[41] within the Deformation Quantization approach[42], but we prefer here to not define the full character of this dot product.

Furthermore, since, due to the radial ordering $\mathbb{R}$ procedure (which for the Green functions in the conformal framework replaces the ordinary T-ordering), this product is intrinsically not commutative, this procedure lays down the foundations for a non-commutative space.

So the Laguerre-Forsyth frame regains its role within the domain of the non-commutative coordinates, giving a possible quantum explanation of this property.

5 Conclusions

The non commutative geometry on coordinates have been introduced in Physics long time ago [43] [44], and provides solutions, at the quantum level, for Field Theory, Elementary Particle Physics and many other branches whose amazing beauty leads to the conclusion that should be a pity to discard this possibility. The commutative limit is reached as a classical limit approximation, and only in this sense the so many success of the centuries-old commutative geometry within the ordinary Mathematics and Physics can be fully accepted.

To recover both the solutions as the consequence of a dynamical phenomenon could be a good compromise [45].

Our operative treatment in any arbitrary dimension, provides an alternative description with no strong magnetic background [46], or M-theory [47], but just assumes a geometrical background only.

We hope that this could be an appreciated suggestion.

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